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Method to modify random matrix theory using short-time behavior in chaotic systems

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We discuss a modification to random matrix theory (RMT) eigenstate statistics that systematically takes into account the nonuniversal short-time behavior of chaotic systems. The method avoids diagonalization of the Hamiltonian, instead requiring only knowledge of short-time dynamics for a chaotic system or ensemble of similar systems. Standard RMT and semiclassical predictions are recovered in the limits of zero Ehrenfest time and infinite Heisenberg time, respectively. As examples, we discuss wave-function autocorrelations and cross correlations and show how the approach leads to a significant improvement in the accuracy for simple chaotic systems where comparison can be made with brute-force diagonalization.

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The statistical structure of chaotic wave functions has been a key topic of investigation from the early history of quantum chaos and wave chaos physics, and its study is essential for improved understanding of resonances, transport, and long-time dynamics in nonintegrable systems [1]. Random matrix theory (RMT) [2], which serves as a zeroth-order approximation for wave-function statistics in the absence of integrability, describes a statistical ensemble of Hamiltonians having no preferred basis. Within RMT, eigenstates are simply random vectors either in the full Hilbert space or in the subspace given by energy and other conservation laws. For a quantum particle in a slowly varying potential, a wave function then behaves locally like a random superposition of plane waves of fixed wave number, as discussed by Berry [3].

The related Bohigas-Giannoni-Schmit conjecture [4], which states that the spectra of individual classically chaotic systems also obey RMT statistics in the semiclassical limit, is similarly well established experimentally and numerically. Recently, significant progress has been made in deriving this result analytically, starting from a periodic orbit representation of the spectrum [5].

As a universal theory, RMT specifically excludes any system-specific behavior. Well-recognized deviations from random wave-function statistics are associated with boundary effects [6,7], finite system size [6], unstable periodic orbits [8], diffusion [1], and two-body random interactions in many-body systems [9,10]. Similar deviations from RMT spectral statistics have also been long recognized [11] and are known to arise from nonuniversal short-time dynamics [12].

Much progress has been made in understanding nonuniversal wave-function behavior in various situations of physical interest, for example, chaotic wave-function correlations in Husimi space associated with classical dynamics [13] and realistic mesoscopic S matrices arising from a diffusive ray picture of wave propagation [14]. Semiclassical methods [15] have proven successful in quantifying the effects on wave functions of boundaries [6,7] and periodic orbit scars [8]. However, the limit implied by semiclassical approximations may not always be achievable or relevant in describing actual experiments. For example, an analysis of electron interaction matrix elements in ballistic quantum dots shows that, even for thousands of electrons in the dot, several statistical quantities of interest typically exceed random wave predictions by a factor of 3 or more; for other quantities the random wave model fails even to predict the correct sign [16] (see also [17]).

In some situations, e.g., [16], brute force diagonalization of the Hamiltonian may be used to obtain correct statistics for the stationary or long-time behavior; but for very large Hilbert spaces, such as those that arise in many-body situations, diagonalization is often impractical. Even where it "works," diagonalization is unlikely to produce much intuition about the relevant physics and must be repeated for each new Hamiltonian. In fact, individual eigenstates of a chaotic Hamiltonian are highly sensitive to perturbations of the system, particularly for multiparticle systems. The *statistics* of such systems are far more robust and remain accurate for small perturbations.

Here, we present a system and basis-independent means of supplementing RMT with short-time dynamical information that eliminates the need for diagonalization of the Hamiltonian, while providing a greatly improved accuracy over RMT and semiclassical methods for finite systems with a finite Ehrenfest time. Instead of treating RMT and nonuniversal short-time behavior separately, we show that they can be naturally combined to produce useful quantitative predictions about wave-function statistics in realistic nonintegrable systems.

To enable direct comparison with RMT, let us consider fully chaotic (ballistic or diffusive) dynamics without symmetry on an N-dimensional Hilbert space with eigenstates $|\xi\rangle$. To avoid ambiguities in the definition of $|\xi\rangle$, we assume a nondegenerate spectrum. Typical quantities of interest, then, are functions of the amplitudes $\langle a|\xi\rangle$ for any physically motivated basis state $|a\rangle$, which may be a position or a momentum state, a Slater determinant, or more generally an eigenstate of some zeroth-order Hamiltonian. With the normalization $\Sigma_{\xi=1}^{N}|\langle a|\xi\rangle|^2=1$, the simplest and first nontrivial moment of these amplitudes is given by the local inverse participation ratio (IPR), which measures the degree of localization at $|a\rangle$,

$$P^{aa}=N\sum_{\xi=1}^{N}|\langle a|\xi\rangle|^4=N\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt|\langle a|a(t)\rangle|^2, \qquad (1)$$

varying from $P^{aa}=1$ in the case of perfect ergodicity to $P^{aa}=N$ for perfect localization. For two arbitrary states we have

$$P^{ab} = N\sum_{\xi=1}^{N} |\langle a|\xi\rangle|^2 |\langle b|\xi\rangle|^2 = N \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt |\langle a|b(t)\rangle|^2. \tag{2}$$

Obviously, higher-order moments and in general the entire joint distribution of the eigenstate intensities may be considered (e.g., [18]). We may also relax the requirement that only pure states such as $|a\rangle\langle a|$ act as probes and instead measure the structure of chaotic eigenstates using any desired self-adjoint operator $\hat{\alpha}$ [19]. Operator probes (of phase-space size greater than or smaller than $\hbar$) will, for example, be particularly helpful in the study of hierarchical eigenstates in a mixed chaotic-regular phase space [20]. Again, without loss of generality we may adopt the normalization Tr $\hat{\alpha}=1$. Equation (2) becomes

$$P^{\alpha\beta} = N\sum_{\xi=1}^{N} \langle \xi|\hat{\alpha}|\xi\rangle\langle \xi|\hat{\beta}|\xi\rangle = N \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt \ \mathrm{Tr}\ \hat{\alpha}\hat{\beta}(t), \tag{3}$$

with the autocorrelation $P^{\alpha\alpha}$ as an obvious special case.

In the semiclassical limit $N\to\infty$, averages of expressions such as (1)–(3) may be obtained using short-time dynamics; specifically, for discrete-time dynamics we have

$$\overline{P^{ab}} \approx \overline{\sum_{-\tau}^{\tau} P^{ab}(t)}, \tag{4}$$

where

$$P^{ab}(t) = |\langle a|b(t)\rangle|^2 + \langle a|a(t)\rangle\langle b(t)|b\rangle, \tag{5}$$

for any two states $|a\rangle$ and $|b\rangle$ (identical, overlapping, or orthogonal) [18]. Here and in the following, $\overline{\cdots}$ indicates an ensemble average. If desired, the ensemble may be selected so that all realizations possess the same short-time dynamics $P^{ab}(t)$, in which case the average on the right-hand side of Eq. (4) is superfluous. The cutoff time τ must be larger than the ballistic or diffusive Thouless time (i.e., the time required for an initial state to spread over the available space, before which universal dynamics is not possible) [21] and short compared to the Heisenberg time, which scales with N. No distinction is made in Eq. (4) between nonuniversal short-time revivals that indicate deviations from RMT in the eigenstate statistics and the $O(1/N)$ short-time revivals that are present already in the context of RMT. As a result, Eq. (4) systematically overestimates corrections to RMT and violates probability conservation $\Sigma_b P^{ab}=1$ given a complete basis $|b\rangle$ for any $\tau>0$, with the violations growing linearly as τ/N.

We now notice that the problematic aspects of Eq. (4) for finite system size N can be eliminated by introducing a τ- and $\langle a|b\rangle$-dependent prefactor

$$\overline{P^{ab}} \approx C_N^{\langle a|b\rangle}(\tau)\overline{\int_{-\tau}^{\tau} dt\ P^{ab}(t)}, \tag{6}$$

where in particular $C_N^{\langle a|b\rangle}(\tau)=N/4\tau$ converges to the exact answer as $\tau\to\infty$. To fix $C_N^{\langle a|b\rangle}$ in general, we apply RMT to Eq. (6) and obtain

$$\overline{P^{ab}} \approx \overline{P^{ab}_{\mathrm{RMT}}}\frac{\overline{\int_{-\tau}^{\tau} dt\ P^{ab}(t)}}{\overline{\int_{-\tau}^{\tau} dt\ P^{ab}_{\mathrm{RMT}}(t)}}. \tag{7}$$

Equation (7) and its natural extensions to higher-order moments [e.g., $\overline{(P^{ab})^n}$] and operator expectation values (e.g., $\overline{P^{\alpha\beta}}$) are a key result of this Rapid Communication. Reassuringly, Eq. (7) yields exact results in three limits of interest: (i) the RMT limit of vanishing Thouless time, where the dynamics is universal at all times down to $t=0$, $P^{ab}(t)=P^{ab}_{\mathrm{RMT}}(t)$, and thus $P^{ab}=P^{ab}_{\mathrm{RMT}}$; (ii) the semiclassical limit $N/\tau\to\infty$, where we recover Eq. (4); and (iii) the limit where an infinite amount of dynamical data is available as input, $\tau\to\infty$. More importantly, we will see in the examples below that Eq. (7) and its extensions provide reliable approximations to exact diagonalization in situations far from any such limit, i.e., for finite-size systems far from universality, and where the only input is short-time dynamics on the scale of a Lyapunov time.

Short-time overlaps $P^{ab}(t)$ needed as input to Eq. (7) may sometimes be known analytically, as in the case of periodic orbit scars, while in more general situations the short-time dynamics for a given system of interest is easily obtainable numerically to any desired time scale τ. The RMT factors in Eq. (7) and its generalizations may be treated entirely analytically. For example, for arbitrary $|a\rangle$ and $|b\rangle$ we have standard results in the absence of time-reversal symmetry, i.e., for the Gaussian unitary ensemble (GUE) or circular unitary ensemble (CUE),

$$\overline{P^{ab}_{\mathrm{RMT}}} = \frac{N}{N+1}(1+|\langle a|b\rangle|^2), \tag{8}$$

while for a general self-adjoint operator $\hat{\alpha}$ we obtain

$$\overline{P^{\alpha\alpha}_{\mathrm{RMT}}} = \frac{N}{N+1}\left(2\sum_i A_i^2 + \sum_{i\neq j} A_i A_j\right), \tag{9}$$

where A_i are the eigenvalues of $\hat{\alpha}$ ($\Sigma_i A_i=1$).

Similarly, RMT dynamical overlaps may be expressed exactly using RMT eigenstate statistics and the RMT spectral form factor, e.g.,

$$\overline{P^{ab}_{\mathrm{RMT}}(t)} = \frac{2}{N}\overline{P^{ab}_{\mathrm{RMT}}} + \sum_{\xi\neq\xi'}(\overline{e^{i(E_{\xi'}-E_{\xi})t}})_{\mathrm{RMT}}(\overline{|\langle a|\xi\rangle|^2|\langle b|\xi'\rangle|^2} + \overline{\langle a|\xi\rangle\langle\xi|b\rangle\langle\xi'|a\rangle\langle b|\xi'\rangle})_{\mathrm{RMT}}. \tag{10}$$

For discrete-time dynamics, described by the CUE ensemble—which will be relevant for the numerical examples below—we have

$$\overline{P^{ab}_{\rm RMT}(t)} = (1+|\langle a|b\rangle|^2) \times \begin{cases} 1, & \text{for } t=0 \\ \dfrac{1+t/N}{N+1}, & \text{for } 1\le |t| \le N \\ \dfrac{2}{N+1}, & \text{for } |t|>N, \end{cases} \tag{11}$$

and analogous results for self-adjoint operators are obtained by spectral decomposition, as in Eq. (9).

Thus, eigenstate statistics for a chaotic system or ensemble of systems may be unambiguously obtained without diagonalization, as in Eq. (7), by combining exact RMT results with easily obtainable short-time dynamical information for the system or ensemble of interest.

We now discuss a few illustrative examples, using as our model the paradigmatic example of a quantized periodically kicked Hamiltonian [22] $H(q,p,t)=T(p)+V(q)\Sigma_{n=-\infty}^{\infty}\delta(t-n)$ on the compact phase space $(q,p)\in[-1/2,1/2)^2$. The kinetic and the potential terms are chosen to produce a fully chaotic map (perturbed cat map [23])

$$T(p) = \frac{m}{2}p^2 + \frac{K}{4\pi^2}\cos(2\pi p) + t(p), \tag{12}$$

$$V(q) = -\frac{m}{2}q^2 - \frac{K}{4\pi^2}\cos(2\pi q) + v(q), \tag{13}$$

where m and K control the chaoticity of the system: the dynamics is fully chaotic for $m>|K|$ and the instability exponent of the shortest periodic orbit at $q=p=0$ is $\lambda_0=\cosh^{-1}[1+(m-K)^2/2]\approx m-K$ for $m-K\ll 1$. To break time-reversal and parity symmetries, and also allow for ensemble averaging of the statistics, we have added the functions $t(p)$ and $v(q)$, which are random within a small region near the edges of the phase space ($|p|>1/2-\Delta$ and $|q|>1/2-\Delta$) and zero elsewhere. In the following, we set $\Delta=0.1$, but the results have no significant dependence on Δ.

We begin by considering the IPR $P^{\alpha\alpha}$, where $\hat{\alpha}$ is the Weyl transform of a Gaussian distribution $\rho(q,p)\sim e^{-q^2/\sigma_q^2-p^2/\sigma_p^2}$ centered on the periodic orbit. We define $s=\sigma_q\sigma_p/\hbar$. Then in the special case $s=1$, $\hat{\alpha}$ is a projection onto a minimum uncertainty Gaussian wave packet, while more generally $\hat{\alpha}$ represents a mixed initial state. Typical results are shown in Fig. 1, where the dynamical prediction of Eq. (7) for several values of the cutoff time τ is compared with exact values obtained by brute-force diagonalization. The dynamical prediction begins at the RMT limit for $\tau=0$, as it must, and quickly converges to the exact stationary answer at two or three Lyapunov times. Figure 2 illustrates the relationship between the exact IPR, the dynamical prediction, and the limiting RMT and semiclassical approximations, as the system size N is varied. Here, we note significant deviations from the semiclassical answer even when N takes values of 100 or greater; these deviations are well reproduced in the dynamical calculation.

As another example, we consider wave-function intensity correlations $\overline{P^{ab}}$ for position states $|a\rangle,|b\rangle$. Since $(1/N^2)\Sigma_{a,b=1}^{N}\overline{P^{ab}}=1$ is given by wave-function normalization

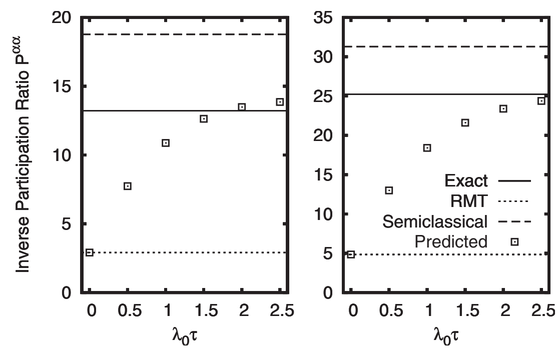

FIG. 1. The IPR $P^{\alpha\alpha}$ for a Gaussian distribution centered on a short periodic orbit with instability exponent $\lambda_0=0.5$ is computed by direct diagonalization and compared with the short-time dynamical prediction given by Eq. (7). Here, the system size is $N=32$ and the Gaussian distribution has a size $s=0.5$ (left panel) or $s=0.25$ (right panel). Convergence to the exact result is observed when the dynamical calculation includes information about times τ up to 2 or 3 in units of the local Lyapunov exponent λ_0. The RMT value $P^{\alpha\alpha}_{\rm RMT}=(1+s^{-1})N/(N+1)$ and the semiclassical result $P^{\alpha\alpha}_{\rm SC}=(1+s^{-1})[N/(N+1)]\Sigma_{t=-\infty}^{\infty}\mathrm{sech}(\lambda_0 t)$ are shown for comparison. All quantities here and in subsequent figures are dimensionless.

when $\tau\to\infty$, we focus on the first interesting moment: the variance

$$W = \frac{1}{N^2}\sum_{a,b}\overline{(P^{ab})^2} - 1. \tag{14}$$

W is a simple measure of nonuniformity in infinite-time transport [24]. Interchanging the roles of the eigenstates and basis states, W may be equivalently written as the variance of the interaction matrix elements $P^{\xi\xi'}=N\Sigma_{a=1}^{N}|\langle a|\xi\rangle|^2|\langle a|\xi'\rangle|^2$ between eigenstates $|\xi\rangle$ and $|\xi'\rangle$, i.e., $W=(1/N^2)\Sigma_{\xi,\xi'}(P^{\xi\xi'})^2-1$. The statistics of such interaction matrix elements in chaotic systems frequently appear in applications ranging from quantum dot conductance in the

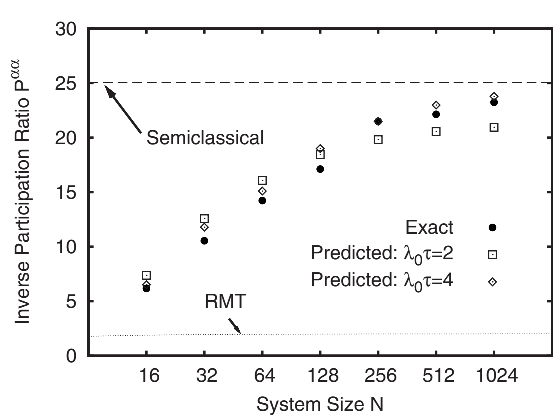

FIG. 2. The IPR for a pure Gaussian wave packet ($s=1$) is computed exactly and compared with the dynamical prediction of Eq. (7) using dynamical information up to times $\tau=2\lambda_0^{-1}$ and $4\lambda_0^{-1}$, where $\lambda_0=0.25$ is the local Lyapunov exponent. Results are shown for various values of the system size N. The semiclassical and RMT limits are also shown for comparison (see Fig. 1 caption).

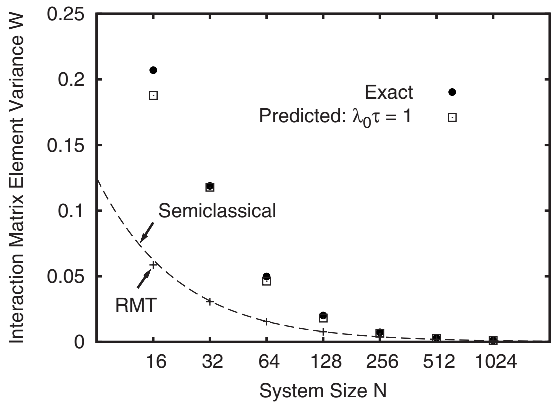

FIG. 3. The interaction matrix element variance W is computed exactly [Eq. (14)] and compared with the short-time prediction (15), with $\tau\lambda_0=1$. Here, $\lambda_0=0.125$. The RMT result and the semiclassical limit $W_{SC}=1/N$ are also shown for comparison.

Coulomb blockade regime [16] to controlling directional emission properties in microcavity lasers [25].

We again combine short-time dynamics and RMT to calculate the variance of the interaction matrix elements, similarly to Eq. (7), as

$$\overline{(P^{ab})^2} \approx \overline{(P^{ab}_{\mathrm{RMT}})^2}\,\frac{\overline{\left(\int_{-\tau}^{\tau} dt\ P^{ab}(t)\right)^2}}{\overline{\left(\int_{-\tau}^{\tau} dt\ P^{ab}_{\mathrm{RMT}}(t)\right)^2}}. \qquad (15)$$

We note here that the intensity correlators P^{ab} predicted by Eq. (7) are not guaranteed to satisfy the normalization condition $(1/N^2)\Sigma_{a,b=1}^{N}\overline{P^{ab}}=1$ unless $\tau\to\infty$, whereas this normalization always holds for the exact correlators. In order to predict W, we already need the knowledge of the short-time dynamics $P^{ab}(t)$ for every pair of initial and final states $|a\rangle,|b\rangle$; therefore, with little added computational effort we may achieve exact normalization and further improve the convergence with τ, simply by rescaling $\overline{(P^{ab})^2}\to\overline{(P^{ab})^2}/[(1/N^2)\Sigma_{a',b'=1}^{N}\overline{P^{a'b'}}]^2$.

Figure 3 shows that the semiclassical and the RMT predictions are very similar for the system we consider here, and both deviate significantly from the exact results for finite N. Our method, including rescaling, converges toward the exact answer very quickly, on the order of one Lyapunov time, even where the RMT prediction is off by a factor of 2 or 3.

We have developed a method that improves on RMT eigenstate statistics for chaotic systems by systematically incorporating short-time dynamics. The method is conceptually appealing, computationally simpler than brute-force diagonalization, and significantly more accurate than the RMT or the semiclassical limit for realistic systems. The approach can be easily extended to consider symmetry effects (including time-reversal symmetry), mixed phase space [26], and resonance wave-function statistics in open systems.

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